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A New Concept of Stability, M_O-Stability

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A new concept of stability called M_O-stability is defined and used to describe a general type of invariant set and its stability behavior. Criteria for M_O-stability are established by using a Lyapunov-type function to obtain comparison equations. A theorem using two auxiliary functions is proved and an instability result is proved.

1. INTRODUCTION

As is well known, Lyapunov's second method, which has its origin in three simple theorems, is an indispensible tool in the theory of stability [2–5]. This method, together with the theory of differential inequalities, has provided a very flexible mechanism to study not only stability theory but also other qualitative and quantitative properties of solutions of differential equations [4]. Stability in the sense of Lyapunov investigates the stability properties of invariant sets. Since in many concrete problems, such as adaptive control systems, one needs to consider the stability of sets which are not invariant, the notion of eventual stability was introduced to deal with such situations [4]. It was subsequently recognized that although the set which is eventually stable is not invariant in the usual sense, it is so in the asymptotic sense [4]. This observation led to a new concept of asymptotically invariant sets, which forms a special subclass of invariant sets, and the discussions of their stability properties [4].

We shall introduce a new concept of stability called M_O-stability to describe a very general type of invariant set and its stability behavior. This notion has naturally led us to consider the initial values on surfaces that crucially depend on initial time and also to introduce different topologies in the definition of stability of M_O-invariant sets.

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2. Preliminary Results

We consider the generalized initial value problem

$$x' = f(t, x), \qquad x(t_0) = \Psi(t_0, x^*), \qquad t_0 \geqslant 0 \tag{1}$$

where f, $\Psi \in C[R_+ \times R^n, R^n]$. We assume f smooth enough to ensure existence of solutions of (1).

We need the following notation before we proceed further [1]. By $M(R_+, R^n)$ we denote the space of all measurable mappings from R_+ to R^n such that $x \in M$ if and only if $x(t)$ is locally integrable on R_+ and

$$\sup_{t>0} \int_t^{t+1} \|x(s)\| \, ds < \infty.$$

Now denote by $M_O(R_+, R^n)$ the subspace of $M(R_+, R^n)$ consisting of all $x(t)$ such that

$$\int_t^{t+1} \|x(s)\| \, ds \to 0 \qquad \text{as} \quad t \to \infty.$$

The set $S(M_O, \varepsilon)$ is the subset of $M(R_+, R^n)$ defined by

$$S(M_O, \varepsilon) = \left\{ x \in M: \limsup_{t \to \infty} \int_t^{t+1} \|x(s)\| \, ds \leqslant \varepsilon \right\}.$$

By this we mean for each $\varepsilon > 0$ there exists a $\tau(\varepsilon) > 0$ with the property that $\tau(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that

$$\int_t^{t+1} \|x(s)\| \, ds < \varepsilon, \qquad t > \tau(\varepsilon).$$

We now give the definitions for M_O-invariant set and the various types of M_O-stability. As usual $x(t, s, \psi(s, x^*))$, $t \geqslant s$, represents a solution to (1) which starts at $(s, \psi(s, x^*))$.

Definition 1. Let $A \subset R^n$. A is M_O-invariant with respect to the system (1) if whenever $x^* \in A$ and $\psi(s, x^*) \in M_O$, then $x(\cdot, s, \psi(s, x^*)) \in M_O$.

Definition 2. The set A is said to be with respect to the system (1)

(M_1) M_O-equistable if for each $\varepsilon > 0$, there exists $\tau_1(\varepsilon)$, $\tau_1(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and a $\delta_1(t_0, \varepsilon)$, $\delta_2(t_0, \varepsilon)$ such that

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds < \varepsilon, \qquad t \geqslant t_0 + 1,$$

provided $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\| \, ds < \delta_2$; $t_0 \geqslant \tau_1(\varepsilon)$;

(M_2) M_O-uniformly stable if δ_1 and δ_2 in (M_1) are independent of t_0;

(M_3) M_O-quasi-equi-asymptotically stable if for every $\varepsilon > 0$, there exist positive numbers $\delta_{1_0}(t_0)$, $\delta_{2_0}(t_0)$, τ_0, and $T(t_0, \varepsilon)$ such that

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds < \varepsilon, \qquad t \geqslant t_0 + 1 + T(t_0, \varepsilon), \, t_0 \geqslant \tau_0,$$

provided $x^* \in S(A, \delta_{1_0})$ and $\psi(s, x^*) \in S(M_0, \delta_{2_0})$;

(M_4) M_O-quasi uniformly asymptotically stable if the numbers δ_{1_0}, δ_{2_0}, T in (M_3) are independent of t_0;

(M_5) M_O-equi asymptotically stable if (M_1) and (M_3) hold;

(M_6) M_O-uniformly asymptotically stable if (M_2) and (M_4) hold;

(M_7) M_O-unstable if (M_1) does not hold.

Remark. The notions (M_1), (M_2), (M_5), and (M_6) relative to A imply that A is M_O-invariant.

Consider the example

$$x' = e^{-t}, \qquad x(t_0) = \psi(t_0, x^*), \qquad t_0 \geqslant 0$$

where $\psi(s, x^*) = x^* + 1/s$. The solution is $x(t, s, \psi(s, x^*)) = x^* + 1/s + e^{-s} - e^{-t}$. It is clear that the set $x = 0$ is M_O-uniformly stable. For this example $x = 0$ is also eventually uniformly stable. If, on the other hand, we choose $\psi(s, x^*) = x^* + \lambda(s)$ where λ: $[0, \infty) \to R$ is a C^1 function coinciding with e^{-t} except at some peaks where it reaches the value 1. There is one peak for each integer value of t and the width of the peak corresponding to abscissa n is smaller than $(\frac{1}{2})^n$ [6]. Then $x = 0$ is not eventually stable, but it is M_O-uniformly stable. This example shows that stability depends on the initial values also.

One can also consider the example

$$x' = -\lambda'(t), \qquad x(t_0) = x^* \tag{2}$$

where λ is defined by

$$\begin{aligned} \lambda(t) &= n, && t = n, \text{ integer} \\ &= 2n^4(t-n) + n, && n - 1/2n^3 < t < n \\ &= -2n^4(t-n) + n, && n < t < n + 1/2n^3 \\ &= 0, && \text{all other } t \geqslant 0. \end{aligned}$$

Then $\lambda'(t)$ exists except on a set of measure zero. Considering only positive solutions to (2) we obtain

$$x(t, s, \psi(s, x^*)) = x^* + \lambda(t_0) - \lambda(t) \leqslant x^* + \lambda(t_0).$$

The set $x=0$ is M_O-uniformly stable since $\int_{t_0}^{t_0+1} \lambda(s)\,ds$ is at most $1/2n^2$ for $n \in [t_0, t_0+1]$. But $x=0$ is not eventually uniformly stable since $\lambda(t_0)$ does not approach zero as $t_0 \to \infty$.

It is convenient to introduce certain classes of monotone functions.

DEFINITION 3. A function a is said to belong to

(i) the class K if $a \in C[R_+, R_+]$, $a(0)=0$, a is strictly monotone increasing in r;

(ii) the class KC if $a \in K$ and a is convex;

(iii) the class P if $a \in C[R_+ \times R_+, R_+]$ and a has the property that given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$a(s, \|\psi(s, x^*)\|) \in S(M_0, \varepsilon),$$

provided $\psi(s, x^*) \in S(M_0, \delta)$.

The following inequality is used [7].

LEMMA 1 (Jensen Inequality). *Let ϕ be a convex function and f integrable, then*

$$\phi\left(\int f(t)\,dt\right) \leqslant \int \phi(f(t))\,dt.$$

Consider the comparison equation

$$u' = g(t, u), \qquad u(t_0) = \phi(t_0, u^*), \qquad t_0 \geqslant 0 \tag{3}$$

where $g \in C[R_+ \times R_+, R]$, $\phi \in C[R_+ \times R_+, R_+]$. We say $u=0$ is M_O-invariant if whenever $\phi(s, 0) \in M_O$, then $u(\cdot, s, \phi(s, 0)) \in M_O$. Now we define concepts analogous to (M_1) to (M_6) as follows. The set $u=0$ is said to be with respect to the differential equation (3)

(M_1^*) M_O-equistable if for each $\varepsilon > 0$, there exists $\tau_1(\varepsilon)$ and $\delta_1(t_0, \varepsilon)$, $\delta_2(t_0, \varepsilon)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*))\,ds < \varepsilon, \qquad t \geqslant t_0 + 1$$

provided $u^* < \delta_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\,ds < \delta_2$, $t_0 \geqslant \tau_1(\varepsilon)$.

The remaining notions (M_2^*)–(M_6^*) corresponding to (M_2)–(M_6) can be easily formulated.

3. Criteria for M_O-Stability

We will use a Lyapunov-type function $V(t, x)$ to obtain comparison equations. For $V \in C[R_+ \times R^n, R_+]$ we define the function [4]

$$D^+ V(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, x+hf(t, x)) - V(t, x)]$$

for $(t, x) \in R_+ \times R^n$.

We will only present results relative to uniform concepts. Based on these one can construct the proofs of the other cases.

THEOREM 1. *Assume there exists functions $V(t, x)$ and $g(t, u)$ satisfying the following conditions*:

(i) $g \in C[R_+ \times R_+, R]$;

(ii) $V \in C[R_+ \times R^n, R_+]$, *$V(t, x)$ is locally Lipschitzian in x*;

(iii) $b(\|x\|) \leqslant V(t, x) \leqslant a(t, \|x\|)$ *where $a \in P$ and $b \in KC$*;

(iv) *for $(t, x) \in R_+ \times R^n$*

$$D^+ V(t, x) \leqslant g(t, V(t, x)).$$

Then (M_2^*) implies (M_2).

Proof. Let $\varepsilon > 0$. Using (M_2^*), there exists $\delta_1^*(\varepsilon)$, $\delta_2^*(\varepsilon)$, and $\tau_1(\varepsilon)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*))\, ds < b(\varepsilon), \qquad t \geqslant t_0 + 1,\ t_0 \geqslant \tau_1(\varepsilon)$$

provided $u^* < \delta_1^*$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\, ds < \delta_2^*$. By (iii) and definition of A, we can find $\hat{\delta}_1(\varepsilon)$, $\delta_2(\varepsilon)$, and $\tau_2(\varepsilon)$ such that the following inequalities will hold simultaneously:

$$\int_{t_0}^{t_0+1} a(s, \|\psi(s, x^*)\|)\, ds < \delta_2^*, \qquad t_0 \geqslant \tau_2(\varepsilon)$$

and $x^* \in S(A, \hat{\delta}_1)$, $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \delta_2$.

Let $\delta_1(\varepsilon) = \min\{\delta_1^*(\varepsilon), \hat{\delta}_1(\varepsilon)\}$ and $\tau(\varepsilon) = \max\{\tau_1(\varepsilon), \tau_2(\varepsilon)\}$. If we choose x^* such that $x^* \in S(A, \delta_1)$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \delta_2$, then

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds < \varepsilon, \qquad t \geqslant t_0 + 1,\ t_0 \geqslant \tau(\varepsilon).$$

Suppose this is not true. Then there exists $t_1 > t_0 + 1$, $t_0 \geqslant \tau(\varepsilon)$, such that

$$\int_{t_0}^{t_0+1} \|x(t_1, s, \psi(s, x^*))\| \, ds = \varepsilon, \qquad \int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds < \varepsilon$$

$$t_0 + 1 \leqslant t < t_1, \; t_0 \geqslant \tau(\varepsilon).$$

Let $r(t, s, \phi(s, u^*))$ be the maximal solution to the differential equation (3). Then by basic comparison theorem [4],

$$V(t, x(t, s, \psi(s, x^*))) \leqslant r(t, s, \phi(s, u^*)) \tag{4}$$

since $V(s, \psi(s, x^*)) \leqslant a(s, \|\psi(s, x^*)\|) \equiv \phi(s, d(A, x^*)) = \phi(s, u^*)$ letting $u^* = d(A, x^*)$. Using (4) and assumption (iii) we obtain the following contradiction:

$$\begin{aligned} b(\varepsilon) &\leqslant b\left(\int_{t_0}^{t_0+1} \|x(t_1, s, \psi(s, x^*))\| \, ds\right) \leqslant \int_{t_0}^{t_0+1} V(t_1, s, \psi(s, x^*)) \, ds \\ &\leqslant \int_{t_0}^{t_0+1} r(t_1, s, \phi(s, u^*)) \, ds \\ &< b(\varepsilon) \end{aligned}$$

since $u^* < \delta_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*) \, ds < \delta_2^*$. This completes the proof.

COROLLARY 1. *The function* $g(t, u) \equiv -\lambda'(t)$ *where* $\lambda \in M_O$, $\lambda \in C^1[R_+, R]$, *is admissible in Theorem* 1.

COROLLARY 2. *The trivial function* $g(t, u) \equiv 0$ *is also admissible in Theorem* 1.

These corollaries correspond to the first theorem of Lyapunov in Lyapunov stability theory.

We note that stability of an invariant set and the stability of an asymptotically invariant set imply M_O-stability of the invariant set and the converse is not true.

THEOREM 2. *Assume the conditions of Theorem* 1 *hold. Then if the set* $u = 0$ *is* M_O*-uniformly asymptotically stable with respect to* (3), *the set* A *is* M_O*-uniformly asymptotically stable with respect to the system* (1).

Proof. By Theorem 1 A is M_O-uniformly stable; we need to prove (M_4) holds. It follows from (M_4^*) that there exists positive numbers $\delta_{1_0}^*$, $\delta_{2_0}^*$, τ_0^*, and $T(\varepsilon)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*)) \, ds < b(\varepsilon), \qquad t \geqslant t_0 + 1 + T(\varepsilon), \; t_0 \geqslant \tau_0^*$$

provided $u^* < \delta_{1_0}^*$ and $\int_{t_0}^{t_0+1} \phi(s, u^*) \, ds < \delta_{2_0}^*$.

As in the proof of Theorem 1, we can find positive numbers $\hat{\delta}_{1_0}$, δ_{2_0}, $\hat{\tau}_0$ which satisfy the inequalities

$$\int_{t_0}^{t_0+1} a(s, \psi(s, x^*))\, ds < \delta^*_{2_0}, \qquad t_0 \geq \hat{\tau}_0$$

and $x^* \in S(A, \hat{\delta}_{1_0})$, $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \delta_{2_0}$. Then $\hat{\tau}_0$ does not depend on ε since $\delta^*_{2_0}$ is independent of ε. Let $\delta_{1_0} = \min\{\hat{\delta}_{1_0}, \delta^*_{1_0}\}$ and $\tau_0 = \max\{\tau^*_0, \hat{\tau}_0\}$. Then we claim

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds < \varepsilon, \qquad t \geq t_0 + 1 + T(\varepsilon),\ t_0 \geq \tau_0$$

when x^* is chosen so $x^* \in S(A, \delta_{1_0})$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \delta_{2_0}$. If not there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that

$$\int_{t_0}^{t_0+1} \|x(t_k, s, \psi(s, x^*))\|\, ds > \varepsilon.$$

This leads to a contradiction as in the proof of Theorem 1:

$$\begin{aligned} b(\varepsilon) &\leq \int_{t_0}^{t_0+1} V(t_k, x(t_k, s, \psi(s, x^*)))\, ds \\ &\leq \int_{t_0}^{t_0+1} r(t_k, s, \phi(s, u^*))\, ds < b(\varepsilon). \end{aligned}$$

COROLLARY 3. *The conclusion of Theorem 2 remains true if the function $g(t, u) = -g_1(u) + \lambda(t)$ where $g_1 \in K$ and is Lipschitz, $\lambda \in M_O$.*

Proof. We will show that (M_6^*) holds. By Corollary 3.4.2 [4] we have $u = 0$ is uniformly asymptotically stable for $u' = -g_1(u)$. We need the following lemma from [8].

LEMMA 2. *If 0 is uniformly asymptotically stable for $x' = f(t, x)$ and f is Lipschitz, g is absolutely diminishing, then 0 is eventually uniformly asymptotically stable for $x' = f(t, x) + g(t)$.*

Since absolutely diminishing is equivalent to belonging to M_O for a function of one variable, we have the solution $u = 0$ of $g(t, u) = g_1(u) + \lambda(t)$ is eventually uniformly asymptotically stable by Lemma 2. This implies (M_6^*) and our result follows from Theorem 2.

Theorems using two auxiliary functions have been introduced and used in the study of stability theory [6]. The next theorem we give uses two such functions.

THEOREM 3. *Suppose there exists two functions $V(t, x)$ and $W(t, x)$, both of which belong to $C[R_+ \times R^n, R_+]$, satisfying the following conditions*:

$$\text{(i)} \quad b(\|x\|) \leqslant V(t, x) \leqslant a(t, \|x\|)$$

where $a \in P$, $b \in KC$;

$$\text{(ii)} \quad d(\|x\|) \leqslant W(t, x)$$

for $d \in KC$;

$$\text{(iii)} \quad d^+ V(t, x) \leqslant -c(W(t, x))$$

when $c \in K$;

(iv) *$d^+ W(t, x)$ is bounded from below or from above.*

Then set A is M_O-equi-asymptotically stable.

Proof. M_O-equistability follows from Corollary 2. From the proof of Theorem 6.23 [6], $W(t, x(t, s, \psi(s, x^*))) \to 0$ as $t \to \infty$. Using condition (ii) we obtain

$$d\left(\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds\right)$$
$$\leqslant \int_{t_0}^{t_0+1} W(t, x(t, s, \psi(s, x^*))) \, ds \to 0$$

as $t \to \infty$. Since $d \in K$, this implies $\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds \to 0$ as $t \to \infty$. Hence we have M_O-quasi-equi-asymptotic stability.

We shall next prove an instability result.

THEOREM 4. *If there exists a function $V(t, x)$ with the properties*

(i) *$V(t, x) \leqslant b(\|x\|)$ where $b(r) \in K$ and $b(\int \phi(s) \, ds) \geqslant \int b(\phi(s)) \, ds$, that is, $-b$ is convex;*

(ii) *for each $\delta > 0$ and $t_0 \geqslant 0$, there exists x^* such that $d(A, x^*) \leqslant \delta$ and $\int_{t_0}^{t_0+1} V(s, \psi(s, x^*)) \, ds > 0$;*

(iii) *$D^+ V(t, x) \geqslant c(\|x\|)$, $c \in KC$;*

then A is M_O-unstable.

Proof. Assume A is M_O-stable. Then for every $\varepsilon > 0$ there exists $\tau_1(\varepsilon)$ and for $t_0 \geqslant \tau_1(\varepsilon)$ there exists $\delta_1(t_0, \varepsilon) > 0$, $\delta_2(t_0, \varepsilon) > 0$ such that $d(A, x^*) \leqslant \delta_1$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\| \, ds < \delta_2$ implies $\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds < \varepsilon$, $t \geqslant t_0 + 1$. Choose x^* such that $d(A, x^*) \leqslant \delta_1$, $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\| \, ds < \delta_2$, and $\int_{t_0}^{t_0+1} V(s, \psi(s, x^*)) \, ds > 0$.

From $d(A, x^*) \leqslant \delta_1$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \delta_2$, it follows that $\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds < \varepsilon$; hence

$$\begin{aligned}\int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))\, ds &\leqslant \int_{t_0}^{t_0+1} b(\|x(t, s, \psi(s, x^*))\|)\, ds \\ &\leqslant b\left(\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds\right) \\ &< b(\varepsilon). \end{aligned} \tag{5}$$

From condition (iii) it follows that $V(t, x(t, s, \psi(s, x^*)))$ is monotone increasing, hence

$$V(t, x(t, s, \psi(s, x^*))) \geqslant V(s, \psi(s, x^*))$$

and

$$\int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))\, ds \geqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds > 0.$$

Therefore, $b(\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds) \geqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds$ and $\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds \geqslant b^{-1}(\int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds)$.

From condition (iii), integrating, we obtain

$$\begin{aligned}&V(t, x(t, t_0, \psi(t_0, x^*))) \\ &\quad \geqslant V(t_0, \psi(t_0, x^*)) + \int_{t_0}^{t} c(\|x(u, t_0, \psi(t_0, x^*))\|)\, du\end{aligned}$$

and

$$\begin{aligned}&\int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))\, ds \\ &\quad \geqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds + \int_{t_0}^{t_0+1} \int_{s}^{t} c(\|x(u, s, \psi(s, x^*))\|)\, du\, ds \\ &\quad \geqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds \\ &\qquad + \int_{t_0+1}^{t} \left(\int_{t_0}^{t_0+1} c\, \|x(u, s, \psi(s, x^*))\|\, ds\right) du \\ &\quad \geqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds \\ &\qquad + c\left[b^{-1}\left(\int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds\right)\right](t - t_0 - 1).\end{aligned}$$

But this means that

$$\operatorname*{limit}_{t \to \infty} \int_{t_0}^{t_0+1} V(t, x(t, s, \psi(s, x^*)))\, ds = +\infty$$

which contradicts (5).

4. Criteria for M_O-Boundedness

Let us first define various M_O-boundedness concepts.

DEFINITION 4. The system (1) is said to be

(MB_1) M_O-equibounded if given $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $t_0 \geqslant 0$, there exists $\beta(t_0, \alpha_1, \alpha_2)$ such that

$$x^* \in S(A, \alpha_1) \qquad \text{and} \qquad \int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \alpha_2$$

imply

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds < \beta(t_0, \alpha_1, \alpha_2), \qquad t \geqslant t_0 + 1;$$

(MB_2) M_O-uniform bounded if the β in (MB_1) is independent of t_0;

(MB_3) M_O-quasi-equi-ultimately bounded if for each $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$, there exist positive numbers N, $\tau(\alpha_1, \alpha_2)$, and $T(t_0, \alpha_1, \alpha_2)$ such that

$$x^* \in S(A, \alpha_1) \qquad \text{and} \qquad \int_{t_0}^{t_0+1} \|\psi(s, x^*)\|\, ds < \alpha_2$$

imply

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\|\, ds < N, \qquad t \geqslant t_0 + 1 + T,\ t_0 \geqslant \tau;$$

(MB_4) M_O-quasi-uniform-ultimately bounded if the T in (MB_3) is independent of t_0;

(MB_5) M_O-equi-ultimately bounded if (MB_1) and (MB_3) hold;

(MB_6) M_O-uniform-ultimately bounded if (MB_2) and (MB_4) hold.

Remark. If the β occurring in (MB_1) has the property that $\beta \to 0$ as $\alpha_1 \to 0$ and $\alpha_2 \to 0$, then the definition (MB_1) implies the definition (M_1).

Analogous to the definitions (MB_1)–(MB_6), we can define the concepts of boundedness with respect to the scalar differential equation (3) and designate them by (MB_1^*) to (MB_6^*).

THEOREM 5. *Assume there exists* $V(t, x)$ *and* $g(t, u)$ *such that*

(i) $g \in C[R_+ \times R_+, R]$;

(ii) $V(t, x) \in C[R_+ \times R^n, R_+]$, $V(t, x)$ *is locally Lipschitzian in* x, *and*

$$b(\|x\|) \leqslant V(t, x) \leqslant a(t, \|x\|)$$

where $b \in KC$, $a \in P$, $b(r) \to \infty$ *as* $r \to \infty$;

(iii) $f \in C[R_+ \times R^n, R^n]$ *and*

$$D^+ V(t, x) \leqslant g(t, V(t, x)), \qquad (t, x) \in R_+ \times R^n.$$

Then (MB_1^*) *implies* (MB_1).

Proof. Given $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$, $t_0 \geqslant 0$. Suppose $x^* \in S(A, \alpha_1)$ and $\int_{t_0}^{t_0+1} \|\psi(s, x^*)\| \, ds < \alpha_2$. Because $a \in P$, there exists $\alpha_3 \geqslant 0$ such that $\int_{t_0}^{t_0+1} a(s, \|\psi(s, x^*)\|) \, ds < \alpha_3$. Using our assumption (MB_1^*), given $\alpha_1 \geqslant 0$ and $\alpha_3 \geqslant 0$, there exists $\beta_1(t_0, \alpha_1, \alpha_2)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*)) \, ds < \beta_1, \qquad t \geqslant t_0 + 1$$

provided $u^* < \alpha_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*) \, ds < \alpha_3$.

Since $b(r) \to \infty$ as $r \to \infty$, we can find a $\beta(t_0, \alpha_1, \alpha_2)$ such that $b(\beta) \geqslant \beta_1$. We can now show (MB_1) holds for β provided $x^* \in S(A, \alpha_1)$ and $\int_{t_0}^{t_0+1} \psi(s, x^*) \, ds < \alpha_2$. Suppose that it does not hold. Then there must exist $t_1 \geqslant t_0 + 1$ such that

$$\int_{t_0}^{t_0+1} \|x(t, s, \psi(s, x^*))\| \, ds \geqslant \beta$$

and $x^* \in S(A, \alpha_1)$, $\int_{t_0}^{t_0+1} \psi(s, x^*) \, ds < \alpha_2$.

By condition (iii) and basic comparison theorem [4], we have

$$V(t, s, \psi(s, x^*)) \leqslant r(t, s, \phi(s, u^*))$$

since $V(s, \psi(s, x^*)) \leqslant a(s, \|\psi(s, x^*)\|) = \phi(s, d(A, x^*)) = \phi(s, u^*)$. We then obtain the following contradiction:

$$\begin{aligned}
\beta_1 \leqslant b(\beta) &\leqslant b\left(\int_{t_0}^{t_0+1} \|x(t_1, s, \psi(s, x^*))\| \, ds\right) \\
&\leqslant \int_{t_0}^{t_0+1} V(t_1, x(t_1, s, \psi(s, x^*))) \, ds \\
&\leqslant \int_{t_0}^{t_0+1} r(t_1, s, \phi(s, u^*)) \, ds < \beta_1
\end{aligned}$$

for some $t_1 > t_0 + 1$ since $u^* < \alpha_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*) \, ds < \alpha_3$.

COROLLARY 4. *The function $g(t, u) \equiv 0$ is admissible in Theorem 5.*

Proof. Given $\alpha_1, \alpha_2 \geqslant 0$, then α_3 exists as in the proof of the theorem. There exists $\beta(t_0, \alpha_1, \alpha_2)$ such that $b(\beta) \geqslant \alpha_3$ and our contradiction becomes

$$\alpha_3 \leqslant \int_{t_0}^{t_0+1} V(t_1, x(t_1, s, \psi(s, x^*)))\, ds \leqslant \int_{t_0}^{t_0+1} V(s, \psi(s, x^*))\, ds$$
$$\leqslant \int_{t_0}^{t_0+1} a(s, \|\psi(s, x^*)\|)\, ds < \alpha_3.$$

THEOREM 6. *Under the assumptions of Theorem 5 (MB_3^*) implies (MB_3) holds.*

Proof. Given $\alpha_1 \geqslant 0$ and $\alpha_2 \geqslant 0$. There exists $\alpha_3(\alpha_2)$ and $\tau_1(\alpha_1, \alpha_2)$ such that $\int_{t_0}^{t_0+1} a(s, \|\psi(s, x^*)\|)\, ds < \alpha_3$ provided $\psi(s, x^*) \in S(M_0, \alpha_2)$ and $t_0 \geqslant \tau_1(\alpha_1, \alpha_2)$. Using (MB_3^*) there exist N_1, $\tau_2(\alpha_1, \alpha_2)$, and $T(t_0, \alpha_1, \alpha_2)$ such that

$$\int_{t_0}^{t_0+1} u(t, s, \phi(s, u^*))\, ds < N_1, \qquad t \geqslant t_0 + 1 + T,\ t_0 \geqslant \tau_2$$

provided $u_0 < \alpha_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\, ds < \alpha_3$.

Let $\tau(\alpha_1, \alpha_2) = \max\{\tau_1(\alpha_1, \alpha_2), \tau_2(\alpha_1, \alpha_2)\}$. Since $b(r) \to \infty$ as $r \to \infty$ there exists N such that $b(N) \geqslant N_1$. We claim for this N, $\tau(\alpha_1, \alpha_2)$, and $T(t_0, \alpha_1, \alpha_2)$ that our result holds. If not there exists a sequence $\{t_k\}$, $t_k \geqslant t_0 + 1 + T$, $t_k \to \infty$ as $k \to \infty$, such that for $x^* \in S(A, \alpha_1)$ and $\psi(s, x^*) \in S(M_0, \alpha_2)$ we have

$$\int_{t_0}^{t_0+1} \|x(t_k, s, \psi(s, x^*))\|\, ds \geqslant N, \qquad t_0 \geqslant \tau.$$

We then have the following contradiction:

$$N_1 \leqslant b(N) \leqslant b\left(\int_{t_0}^{t_0+1} \|x(t_k, s, \psi(s, x^*))\|\, ds\right)$$
$$\leqslant \int_{t_0}^{t_0+1} V(t_k, x(t_k, s, \psi(s, x^*)))\, ds$$
$$\leqslant \int_{t_0}^{t_0+1} r(t_k, s, \phi(s, u^*))\, ds < N_1$$

where $u^* = d(A, x^*) < \alpha_1$ and $\int_{t_0}^{t_0+1} \phi(s, u^*)\, ds = \int_{t_0}^{t_0+1} a(s, \|\psi(s, x^*)\|)\, ds < \alpha_3$. Our proof is complete.

THEOREM 7. *Under the assumptions of Theorem* 5 (MB_5^*) *implies* (MB_5).

Proof. This theorem is proved by combining the proofs of Theorem 5 and Theorem 6.

We have presented the proofs here for equi-concepts. The proofs for uniform cases are similar.

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